q-deformed $su(2|2)$ boundary S-matrices via the ZF algebra

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ABSTRACT: Beisert and Koroteev have recently found a bulk S-matrix corresponding to a q-deformation of the centrally-extended $su(2|2)$ algebra of AdS/CFT. We formulate the associated Zamolodchikov-Faddeev algebra, using which we derive factorizable boundary S-matrices that generalize those of Hofman and Maldacena.

Contents

1. Introduction

In investigations of integrability in AdS/CFT (for reviews, see for example [1]), a centrally-extended $su(2|2)$ algebra (more precisely, two such copies) has emerged as a key symmetry: it is the residual symmetry algebra of both planar $\mathcal{N} = 4$ super Yang-Mills theory [2] and the gauge-fixed $AdS_5 \times S^5$ superstring sigma model [3]. This symmetry leads directly to a bulk S-matrix [2] for the fundamental excitations [4] which obeys a twisted (dynamical) Yang-Baxter equation. By introducing Zamolodchikov-Faddeev (ZF) operators [5, 6] that have suitable commutation relations with the symmetry generators, a related bulk S-matrix can be derived [7] which obeys the standard Yang-Baxter equation. These S-matrices have been used to prove [2, 8, 9] a previously-conjectured set of asymptotic Bethe equations [10] for the spectrum of the gauge/string theory.

Some of these results have been generalized to the case where there is a boundary. Hofman and Maldacena [11] have considered open strings attached to maximal giant gravitons [12] in $AdS_5 \times S^5$. (See also [13 – 15] and references therein.) Based on the residual symmetries, they have derived corresponding boundary S-matrices. By extending the ZF algebra [7] to the boundary case, related boundary S-matrices which obey the standard boundary Yang-Baxter equation [16, 17] have been derived in [18].

A q-deformation of this centrally-extended $su(2|2)$ algebra has recently been considered by Beisert and Koroteev [19]. They derived a corresponding bulk S-matrix, which they related to a deformation [20] of the one-dimensional Hubbard model [21].

In this note, we formulate the ZF algebra associated with this deformed symmetry algebra, using which we derive corresponding factorizable boundary S-matrices. The ZF

formalism is particularly convenient for performing explicit calculations, as the coproduct and braiding relations are encoded in the commutation relations of the ZF operators with the symmetry generators. Using these deformed bulk and boundary S-matrices as inputs into Sklyanin's generalization of the Quantum Inverse Scattering Method for systems with boundaries [22], it should be possible to construct and solve open versions of the deformed Hubbard model. However, even for the undeformed case, this problem remains a challenge.

This paper is organized as follows. In section 2 we recall the definition of the q-deformed centrally-extended $su(2|2)$ algebra [19]. In section 3, we introduce the bulk ZF algebra, and present the commutation relations of the ZF operators with the symmetry generators. As a check on these relations, we use them to recover the Beisert-Koroteev S-matrix. We address boundary scattering in section 4. We begin by determining how $x^{\pm}$ transforms under the reflection $p \mapsto -p$. We then extend the ZF algebra by introducing suitable boundary operators, and proceed to construct q-deformations of the $Y = 0$ and $Z = 0$ giant graviton brane boundary S-matrices of Hofman and Maldacena.

2. The q-deformed algebra

We briefly review here the q-deformed centrally-extended $su(2|2)$ algebra. Following [19], we work in the Chevalley basis, with three Cartan generators h_j, three simple positive roots E_j and three simple negative roots F_j, $j = 1, 2, 3$. The generators E_2, F_2 are fermionic, while the remaining ones are bosonic. The commutators with the Cartan generators are given by

$$[h_j, h_k] = 0, \qquad [h_j, E_k] = A_{jk} E_k, \qquad [h_j, F_k] = -A_{jk} F_k, \qquad (2.1)$$

where A_{jk} is the symmetric Cartan matrix

$$A_{jk} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}. \qquad (2.2)$$

The commutators of E_j with F_j are given by

$$[E_1, F_1] = [h_1]_q, \qquad \{E_2, F_2\} = -[h_2]_q, \qquad [E_3, F_3] = -[h_3]_q, \qquad (2.3)$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \qquad (2.4)$$

and q is the deformation parameter. The remaining mixed commutators vanish,

$$[E_j, F_k] = 0, \qquad j \neq k. \qquad (2.5)$$

The Serre relations are given by

$$[E_1, E_3] = [F_1, F_3] = E_2 E_2 = F_2 F_2 = 0, \qquad (2.6)$$

$$E_1 E_1 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 E_1 E_1 = E_3 E_3 E_2 - (q + q^{-1}) E_3 E_2 E_3 + E_2 E_3 E_3 = 0,$$

$$F_1 F_1 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 F_1 F_1 = F_3 F_3 F_2 - (q + q^{-1}) F_3 F_2 F_3 + F_2 F_3 F_3 = 0.$$

The algebra has three central elements given by

$$C = -\frac{1}{2}h_1 - h_2 - \frac{1}{2}h_3 \,,$$
$$P = E_1 E_2 E_3 E_2 + E_2 E_3 E_2 E_1 + E_3 E_2 E_1 E_2 + E_2 E_1 E_2 E_3 - (q + q^{-1}) E_2 E_1 E_3 E_2 \,,$$
$$K = F_1 F_2 F_3 F_2 + F_2 F_3 F_2 F_1 + F_3 F_2 F_1 F_2 + F_2 F_1 F_2 F_3 - (q + q^{-1}) F_2 F_1 F_3 F_2 \,. \tag{2.7}$$

3. Bulk scattering

We introduce here the bulk ZF algebra, and present the commutation relations of the ZF operators with the symmetry generators. As a check on these relations, we then verify that the Beisert-Koroteev S-matrix can be recovered by demanding that the symmetry generators commute with two-particle scattering.

3.1 Bulk ZF algebra

Following [7, 18], we denote the ZF operators by $A_i^\dagger(p)$, $i = 1, 2, 3, 4$. These operators create asymptotic particle states of momentum p when acting on the vacuum state $|0\rangle$, corresponding to $|\phi^1\rangle, |\phi^2\rangle, |\psi^1\rangle, |\psi^2\rangle$ in [19], respectively. Hence, the first two operators are bosonic, while the last two operators are fermionic. The bulk S-matrix elements $S_{i\,j}^{i'j'}(p_1, p_2)$ are defined by the relation

$$A_i^\dagger(p_1) A_j^\dagger(p_2) = S_{i\,j}^{i'j'}(p_1, p_2) A_{j'}^\dagger(p_2) A_{i'}^\dagger(p_1) \,, \tag{3.1}$$

where summation over repeated indices is always understood. It is convenient to arrange these matrix elements into a 16×16 matrix S as follows,

$$S = S_{i\,j}^{i'j'} e_{i\,i'} \otimes e_{j\,j'} \,, \tag{3.2}$$

where e_{ij} is the usual elementary 4×4 matrix whose (i, j) matrix element is 1, and all others are zero. Associativity of the ZF algebra implies [5] the Yang-Baxter equation,

$$S_{12}(p_1, p_2) S_{13}(p_1, p_3) S_{23}(p_2, p_3) = S_{23}(p_2, p_3) S_{13}(p_1, p_3) S_{12}(p_1, p_2) \,. \tag{3.3}$$

We use the standard convention $S_{12} = S \otimes \mathbb{I}$, $S_{23} = \mathbb{I} \otimes S$, and $S_{13} = \mathcal{P}_{12} S_{23} \mathcal{P}_{12}$, where $\mathcal{P}_{12} = \mathcal{P} \otimes \mathbb{I}$, $\mathcal{P} = e_{i\,j} \otimes e_{j\,i}$ is the permutation matrix, and $\mathbb{I}$ is the four-dimensional identity matrix.

The one-particle states $A_i^\dagger(p)|0\rangle$ must form a fundamental representation of the symmetry algebra (see eq. (2.55) in [19]); and similarly, multi-particle states must form higher (reducible) representations. From these requirements, and the fact that the symmetry generators annihilate the vacuum state, we can abstract the action of the symmetry generators on the ZF operators.

The nontrivial commutators of the Cartan generators with the ZF operators are given by

$$h_1 \, A_1^\dagger(p) = -A_1^\dagger(p) + A_1^\dagger(p) \, h_1 \,, \qquad h_1 \, A_2^\dagger(p) = A_2^\dagger(p) + A_2^\dagger(p) \, h_1 \,,$$

$$h_3 \, A_3^\dagger(p) = -A_3^\dagger(p) + A_3^\dagger(p) \, h_3 \,, \qquad h_3 \, A_4^\dagger(p) = A_4^\dagger(p) + A_4^\dagger(p) \, h_3 \,, \tag{3.4}$$

$$h_2 \, A_1^\dagger(p) = -\left(C - \frac{1}{2}\right) A_1^\dagger(p) + A_1^\dagger(p) \, h_2 \,, \quad h_2 \, A_2^\dagger(p) = -\left(C + \frac{1}{2}\right) A_2^\dagger(p) + A_2^\dagger(p) \, h_2 \,,$$

$$h_2 \, A_3^\dagger(p) = -\left(C - \frac{1}{2}\right) A_3^\dagger(p) + A_3^\dagger(p) \, h_2 \,, \quad h_2 \, A_4^\dagger(p) = -\left(C + \frac{1}{2}\right) A_4^\dagger(p) + A_4^\dagger(p) \, h_2 \,,$$

where $C = C(p)$ denotes the value of the corresponding central charge C (2.7). The remaining such commutators are trivial, $h_j \, A_k^\dagger(p) = A_k^\dagger(p) \, h_j$.

The nontrivial commutators of the bosonic simple roots with the ZF operators are given by

$$E_1 \, A_1^\dagger(p) = q^{1/2} \, A_2^\dagger(p) \, q^{-h_1/2} + q^{-1/2} \, A_1^\dagger(p) \, E_1 \,, \qquad E_1 \, A_2^\dagger(p) = q^{1/2} \, A_2^\dagger(p) \, E_1 \,,$$

$$E_3 \, A_4^\dagger(p) = q^{-1/2} \, A_3^\dagger(p) \, q^{-h_3/2} + q^{1/2} \, A_4^\dagger(p) \, E_3 \,, \qquad E_3 \, A_3^\dagger(p) = q^{-1/2} \, A_3^\dagger(p) \, E_3 \,,$$

$$F_1 \, A_2^\dagger(p) = q^{-1/2} \, A_1^\dagger(p) \, q^{-h_1/2} + q^{1/2} \, A_2^\dagger(p) \, F_1 \,, \qquad F_1 \, A_1^\dagger(p) = q^{-1/2} \, A_1^\dagger(p) \, F_1 \,,$$

$$F_3 \, A_3^\dagger(p) = q^{1/2} \, A_4^\dagger(p) \, q^{-h_3/2} + q^{-1/2} \, A_3^\dagger(p) \, F_3 \,, \qquad F_3 \, A_4^\dagger(p) = q^{1/2} \, A_4^\dagger(p) \, F_3 \,. \tag{3.5}$$

The remaining such commutators are trivial, namely,

$$E_1 \, A_\alpha^\dagger(p) = A_\alpha^\dagger(p) \, E_1 \,, \qquad F_1 \, A_\alpha^\dagger(p) = A_\alpha^\dagger(p) \, F_1 \,, \qquad \alpha = 3, 4 \,,$$

$$E_3 \, A_a^\dagger(p) = A_a^\dagger(p) \, E_3 \,, \qquad F_3 \, A_a^\dagger(p) = A_a^\dagger(p) \, F_3 \,, \qquad a = 1, 2 \,. \tag{3.6}$$

Finally, the commutators of the fermionic generators with the ZF operators are given by

$$E_2 \, A_2^\dagger(p) = e^{-ip/2} \left[a(p) \, A_4^\dagger(p) \, q^{-h_2/2} + q^{-(C+\frac{1}{2})/2} \, A_2^\dagger(p) \, E_2 \right] \,,$$

$$E_2 \, A_3^\dagger(p) = e^{-ip/2} \left[b(p) \, A_1^\dagger(p) \, q^{-h_2/2} - q^{-(C-\frac{1}{2})/2} \, A_3^\dagger(p) \, E_2 \right] \,,$$

$$F_2 \, A_1^\dagger(p) = e^{ip/2} \left[c(p) \, A_3^\dagger(p) \, q^{-h_2/2} + q^{-(C-\frac{1}{2})/2} \, A_1^\dagger(p) \, F_2 \right] \,,$$

$$F_2 \, A_4^\dagger(p) = e^{ip/2} \left[d(p) \, A_2^\dagger(p) \, q^{-h_2/2} - q^{-(C+\frac{1}{2})/2} \, A_4^\dagger(p) \, F_2 \right] \,, \tag{3.7}$$

and

$$E_2 \, A_1^\dagger(p) = e^{-ip/2} q^{-(C-\frac{1}{2})/2} \, A_1^\dagger(p) \, E_2 \,, \qquad E_2 \, A_4^\dagger(p) = -e^{-ip/2} q^{-(C+\frac{1}{2})/2} \, A_4^\dagger(p) \, E_2 \,,$$

$$F_2 \, A_2^\dagger(p) = e^{ip/2} q^{-(C+\frac{1}{2})/2} \, A_2^\dagger(p) \, F_2 \,, \qquad F_2 \, A_3^\dagger(p) = -e^{ip/2} q^{-(C-\frac{1}{2})/2} \, A_3^\dagger(p) \, F_2 \,. \tag{3.8}$$

The one-particle states form a representation of the algebra with $P = e^{-ip} ab$, $K = e^{ip} cd$, provided the functions a, b, c, d obey the constraints [19]

$$ad = \left[C + \frac{1}{2} \right]_q \,, \quad bc = \left[C - \frac{1}{2} \right]_q \,, \tag{3.9}$$

which imply

$$(ad - qbc)(ad - q^{-1}bc) = 1 \,. \tag{3.10}$$

We have verified that the above commutation relations are consistent with the symmetry algebra (2.1)–(2.6). Notice the appearance of the Cartan generators h_j in the commutation relations (3.5), (3.7), which is necessary to implement the nontrivial coproduct. (See, for example, [23].)

A further constraint on the functions a, b, c, d comes from the requirement [19] that the central charges P and K (2.7) commute with two-particle scattering. Indeed, acting with P on both sides of[1]

$$A_1^\dagger(p_1)\,A_1^\dagger(p_2)\,|0\rangle = S_{1\,1}^{1\,1}(p_1, p_2)\,A_1^\dagger(p_2)\,A_1^\dagger(p_1)\,|0\rangle \,, \tag{3.11}$$

we obtain

$$e^{-ip_1}q^{C_2}a_1b_1 + e^{-i(p_1+p_2)}q^{-C_1}a_2b_2 = e^{-ip_2}q^{C_1}a_2b_2 + e^{-i(p_1+p_2)}q^{-C_2}a_1b_1 \,, \tag{3.12}$$

which implies

$$\frac{a_1b_1}{q^{C_1}e^{ip_1} - q^{-C_1}} = \frac{a_2b_2}{q^{C_2}e^{ip_2} - q^{-C_2}} = constant \,. \tag{3.13}$$

Similarly, acting with K on (3.11), we obtain

$$\frac{c_1d_1}{q^{C_1}e^{-ip_1} - q^{-C_1}} = \frac{c_2d_2}{q^{C_2}e^{-ip_2} - q^{-C_2}} = constant \,. \tag{3.14}$$

The constraints (3.13) and (3.14) are satisfied if we set[2]

$$\begin{aligned}
a &= \sqrt{g}\,\gamma\,q^{-C} \,, \\
b &= \sqrt{g}\,\frac{\alpha}{\gamma}\,\frac{1}{x^-}\left(x^- - q^{2C-1}x^+\right) \,, \\
c &= i\sqrt{g}\,\frac{\gamma}{\alpha}\,\frac{q^{-C+\frac{1}{2}}}{x^+} \,, \\
d &= i\sqrt{g}\,\frac{q^{-\frac{1}{2}}}{\gamma}\left(q^{2C+1}x^- - x^+\right) \,,
\end{aligned} \tag{3.15}$$

with

$$e^{ip} = \frac{x^+}{qx^-} \,. \tag{3.16}$$

For simplicity, we henceforth set $\alpha = 1$; but (as in [19]) we leave γ unspecified.

The constraints (3.9) then imply [19]

$$q^{2C} = \frac{1}{q}\left(\frac{1 - ig(q - q^{-1})x^+}{1 - ig(q - q^{-1})x^-}\right) = q\left(\frac{1 + ig(q - q^{-1})/x^+}{1 + ig(q - q^{-1})/x^-}\right) \,. \tag{3.17}$$

These relations in turn imply the quadratic constraint [19]

$$\frac{x^+}{q} + \frac{q}{x^+} - qx^- - \frac{1}{qx^-} + ig(q - q^{-1})\left(\frac{x^+}{qx^-} - \frac{qx^-}{x^+}\right) = \frac{i}{g} \,. \tag{3.18}$$

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[1]The S-matrix elements $S_{1\,1}^{i\,j}$ with $i, j \neq 1$ vanish, as can be seen from (3.19) below.

[2]Our expressions for a and d differ from those in [19] by factors of $q^{\mp C}$. Also, Beisert and Koroteev do not introduce a momentum variable p; instead, they work with $U = e^{ip/2}$.

3.2 Bulk S-matrix

As usual, we can determine the two-particle S-matrix (up to a phase) by demanding that the symmetry generators commute with two-particle scattering. That is, starting from $J A_i^\dagger(p_1) A_j^\dagger(p_2)|0\rangle$ where J is a symmetry generator, and assuming that J annihilates the vacuum state, we arrive at linear combinations of $A_{j'}^\dagger(p_2) A_{i'}^\dagger(p_1)|0\rangle$ in two different ways, by applying the ZF relation (3.1) and the symmetry relations (3.4)–(3.8) in different orders. The consistency condition is a system of linear equations for the S-matrix elements. The result for the nonzero matrix elements is

$$S_{a\,a}^{a\,a} = \mathcal{A}\,, \qquad\qquad\qquad S_{\alpha\,\alpha}^{\alpha\,\alpha} = \mathcal{D}\,,$$

$$S_{a\,b}^{a\,b} = \frac{\mathcal{A} - \mathcal{B}}{q + q^{-1}}\,, \qquad\qquad S_{a\,b}^{b\,a} = \frac{q^{-\epsilon_{ab}}\mathcal{A} + q^{\epsilon_{ab}}\mathcal{B}}{q + q^{-1}}\,,$$

$$S_{\alpha\,\beta}^{\alpha\,\beta} = \frac{\mathcal{D} - \mathcal{E}}{q + q^{-1}}\,, \qquad\qquad S_{\alpha\,\beta}^{\beta\,\alpha} = \frac{q^{-\epsilon_{\alpha\beta}}\mathcal{D} + q^{\epsilon_{\alpha\beta}}\mathcal{E}}{q + q^{-1}}\,,$$

$$S_{a\,b}^{\alpha\,\beta} = q^{(\epsilon_{ab} - \epsilon_{\alpha\beta})/2}\epsilon_{ab}\epsilon^{\alpha\beta} \frac{\mathcal{C}}{q + q^{-1}}\,, \qquad S_{\alpha\,\beta}^{a\,b} = q^{(\epsilon_{\alpha\beta} - \epsilon_{ab})/2}\epsilon^{ab}\epsilon_{\alpha\beta} \frac{\mathcal{F}}{q + q^{-1}}\,,$$

$$S_{a\,\alpha}^{a\,\alpha} = \mathcal{L}\,, \qquad\qquad\qquad S_{\alpha\,a}^{a\,\alpha} = \mathcal{K}\,, \quad S_{\alpha\,a}^{a\,\alpha} = \mathcal{H}\,, \quad S_{a\,\alpha}^{\alpha\,a} = \mathcal{G}\,, \qquad (3.19)$$

where $a, b \in \{1, 2\}$ with $a \neq b$; $\alpha, \beta \in \{3, 4\}$ with $\alpha \neq \beta$; and

$$\mathcal{A} = A_{21}^{BK} = q^{C_2 - C_1} e^{i(p_2 - p_1)/2} \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\,,$$

$$\mathcal{B} = B_{21}^{BK} = q^{C_2 - C_1} e^{i(p_2 - p_1)/2} \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \left(1 - (q + q^{-1})q^{-1} \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} \frac{x_1^- - s(x_2^+)}{x_1^- - s(x_2^-)}\right)\,,$$

$$\mathcal{C} = q^{-(C_1 + C_2 - 1)/2} C_{21}^{BK}$$
$$= (q + q^{-1})igq^{(C_2 - 5C_1 - 2)/2} e^{i(p_2 - 2p_1)/2}\gamma_1\gamma_2 \frac{ig^{-1}x_1^+ - (q - q^{-1})}{x_1^- - s(x_2^-)} \frac{s(x_1^+) - s(x_2^+)}{x_1^- - x_2^+}\,,$$

$$\mathcal{D} = -1\,,$$

$$\mathcal{E} = E_{21}^{BK} = -\left(1 - (q + q^{-1})q^{-2C_1 - 1} e^{-ip_1} \frac{x_1^+ - x_2^+}{x_1^- - x_2^+} \frac{x_1^+ - s(x_2^-)}{x_1^- - s(x_2^-)}\right)\,,$$

$$\mathcal{F} = q^{(C_1 + C_2 - 1)/2} F_{21}^{BK}$$
$$= -(q + q^{-1})igq^{(5C_2 - C_1 - 2)/2} e^{i(2p_2 - p_1)/2} \frac{ig^{-1}x_1^+ - (q - q^{-1})}{x_1^- - s(x_2^-)} \frac{s(x_1^+) - s(x_2^+)}{x_1^- - x_2^+}$$
$$\cdot \frac{1}{1 - g^2(q + q^{-1})^2} \frac{1}{\gamma_1\gamma_2}(x_1^+ - x_1^-)(x_2^+ - x_2^-)\,,$$

$$\mathcal{G} = G_{21}^{BK} = q^{-C_1 - 1/2} e^{-ip_1/2} \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,,$$

$$\mathcal{H} = q^{(C_1 - C_2)/2} H_{21}^{BK} = q^{(C_1 - C_2)/2} \frac{\gamma_2}{\gamma_1} \frac{x_1^+ - x_1^-}{x_1^- - x_2^+}\,,$$

$$\mathcal{K} = q^{-(C_1 - C_2)/2} K_{21}^{BK} = q^{3(C_2 - C_1)/2} e^{i(p_2 - p_1)/2} \frac{\gamma_1}{\gamma_2} \frac{x_2^+ - x_2^-}{x_1^- - x_2^+}\,,$$

$$\mathcal{L} = L_{21}^{BK} = q^{C_2 + 1/2} e^{ip_2/2} \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\,, \qquad (3.20)$$

where A_{21}^{BK}, B_{21}^{BK}, ... denote the amplitudes A_{12}, B_{12}, ... in table 2 of [19], respectively, with labels 1 and 2 interchanged. As already mentioned, we have set the parameter α, as well as the overall scalar factor (denoted by R^0 in [19]), equal to one. The function $s(x)$ is the "antipode map" defined by [19]

$$s(x) = \frac{1 - ig(q - q^{-1})x}{x + ig(q - q^{-1})} \,, \tag{3.21}$$

which has the limit $s(x) \to 1/x$ for $q \to 1$. Our amplitudes $\mathcal{C}$, $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{K}$ evidently have extra factors involving powers of q with respect to the amplitudes of Beisert and Koroteev. However, we have verified with Mathematica that the S-matrix satisfies the Yang-Baxter equation (3.3) even without those extra factors. Hence, these factors can presumably be removed by a suitable gauge transformation.

4. Boundary scattering

A prerequisite to studying boundary scattering is to understand how $x^{\pm}$ transforms under the reflection $p \mapsto -p$. We claim that

$$x^+(-p) = -\frac{1}{s(x^-(p))} \,, \qquad x^-(-p) = -\frac{1}{s(x^+(p))} \,, \tag{4.1}$$

where $s(x)$ is given by (3.21). Indeed, (4.1) has the correct $q \to 1$ limit, namely, $x^{\pm}(-p) = -x^{\mp}(p)$ [11]. Moreover, the momentum relation (3.16) is preserved by this transformation

$$e^{-ip} = \frac{x^+(-p)}{qx^-(-p)} = \frac{s(x^+(p))}{qs(x^-(p))} = \frac{qx^-(p)}{x^+(p)} \,, \tag{4.2}$$

where the final equality is an identity which can be found in appendix A of [19]. Also, the transformation (4.1) preserves the energy,

$$C(-p) = C(p) \,. \tag{4.3}$$

This can easily be seen as follows, starting from (3.17),

$$q^{2C(-p)} = q \left(\frac{1 + ig(q - q^{-1})/x^+(-p)}{1 + ig(q - q^{-1})/x^-(-p)} \right) = q \left(\frac{1 - ig(q - q^{-1})\,s(x^-(p))}{1 - ig(q - q^{-1})\,s(x^+(p))} \right)$$
$$= q \left(\frac{1 + ig(q - q^{-1})/x^+(p)}{1 + ig(q - q^{-1})/x^-(p)} \right) = q^{2C(p)} \,, \tag{4.4}$$

where the first equality on the second line follows from the identity

$$\frac{1 - ig(q - q^{-1})\,s(x)}{1 - ig(q - q^{-1})\,s(y)} = \frac{1 + ig(q - q^{-1})/y}{1 + ig(q - q^{-1})/x} \tag{4.5}$$

which holds for arbitrary values of x and y. Furthermore, the transformation (4.1) preserves the quadratic relation (3.18), since

$$-\frac{1}{qs(x^-)} - qs(x^-) + \frac{q}{s(x^+)} + \frac{s(x^+)}{q} + ig(q - q^{-1}) \left[\frac{s(x^+)}{qs(x^-)} - \frac{qs(x^-)}{s(x^+)} \right] = \frac{i}{g} \,, \tag{4.6}$$

which can be seen most readily from the relation (3.17) and the fact (4.3). Finally, we verify that the transformation (4.1) squares to the identity, by virtue of the identity

$$-s\left(-\frac{1}{s(x)}\right) = \frac{1}{x}\,,\tag{4.7}$$

which holds for arbitrary values of x.

Having determined how $x^\pm$ transforms under reflection, we turn now to the problem of computing boundary S-matrices. Following the approach in [18], we shall extend the ZF algebra (3.1) by introducing suitable boundary operators which create the boundary-theory vacuum state $|0\rangle_B$ from $|0\rangle$ [17]. We shall then proceed, using the commutation relations of the ZF operators with the symmetry generators found in the previous section, to construct q-deformations of the $Y = 0$ and $Z = 0$ giant graviton brane boundary S-matrices of Hofman and Maldacena [11].

4.1 $Y = 0$ giant graviton brane

Since there is no boundary degree of freedom for the $Y = 0$ giant graviton brane, the corresponding boundary operator is a scalar, B. The boundary S-matrix is defined by[3]

$$A_i^\dagger(p)\, B = R_i^{i'}(p)\, A_{i'}^\dagger(-p)\, B\,.\tag{4.8}$$

We arrange the S-matrix elements in the usual way into a matrix $R = R_i^{i'}\, e_{i\,i'}$. Starting from $A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, B$, one can arrive at linear combinations of $A_{i'''}^\dagger(-p_1)\, A_{j'''}^\dagger(-p_2)\, B$ by applying each of the relations (3.1) and (4.8) two times, in two different ways. The consistency condition is the BYBE [16, 17]

$$S_{12}(p_1, p_2)\, R_1(p_1)\, S_{21}(p_2, -p_1)\, R_2(p_2) = R_2(p_2)\, S_{12}(p_1, -p_2)\, R_1(p_1)\, S_{21}(-p_2, -p_1)\,.\tag{4.9}$$

Let us assume that the vacuum state $B|0\rangle$ breaks E_1, F_1, but preserves E_3, F_3. It follows from (3.5) that the boundary S-matrix is diagonal, with matrix elements

$$R_1^1 = r_1\,,\quad R_2^2 = r_2\,,\quad R_3^3 = R_4^4 = r\,.\tag{4.10}$$

Using first (3.7) and then (4.8), we find

$$E_2\, A_2^\dagger(p)\, B|0\rangle = e^{-ip/2}a(p)A_4^\dagger(p)B|0\rangle = e^{-ip/2}a(p)rA_4^\dagger(-p)B|0\rangle\,,\tag{4.11}$$

where we have passed to the second equality using also the assumption that E_2 annihilates the vacuum state. Reversing the order, i.e., using first (4.8) and then (3.7), we obtain

$$E_2\, A_2^\dagger(p)\, B|0\rangle = r_2 E_2\, A_2^\dagger(-p)\, B|0\rangle = r_2 e^{ip/2}a(-p)A_4^\dagger(-p)\, B|0\rangle\,.\tag{4.12}$$

Consistency of the results (4.11) and (4.12) requires

$$\frac{r_2}{r} = e^{-ip}\frac{a(p)}{a(-p)} = e^{-ip}\frac{d(-p)}{d(p)} = e^{-ip}\frac{\gamma(p)}{\gamma(-p)}\,,\tag{4.13}$$

---

[3]We restrict our attention to the *right* boundary S-matrix, since the left boundary S-matrix can be obtained by $p \mapsto -p$ [11, 18].

where, in passing to the second equality, we have used (3.9) and (4.3). Similarly, starting from $E_2 A_3^\dagger(p) B|0\rangle$, we obtain

$$\frac{r_1}{r} = e^{ip}\frac{b(-p)}{b(p)} = e^{ip}\frac{c(p)}{c(-p)} = e^{ip}\frac{x^+(-p)}{x^+(p)}\frac{\gamma(p)}{\gamma(-p)} = -\frac{e^{ip}}{x^+s(x^-)}\frac{\gamma(p)}{\gamma(-p)},\qquad(4.14)$$

where we have used (4.1). The same results are obtained using instead F_2. We conclude that the boundary S-matrix is given (up to a scalar factor) by the diagonal matrix

$$R(p) = \mathrm{diag}(-\frac{e^{ip}}{x^+s(x^-)}\frac{\gamma(p)}{\gamma(-p)}, e^{-ip}\frac{\gamma(p)}{\gamma(-p)}, 1, 1).\qquad(4.15)$$

We have explicitly verified with Mathematica that this matrix satisfies the BYBE (4.9). In the $q \to 1$ limit, (4.15) reduces to the corresponding undeformed boundary S-matrix in [18].

4.2 $Z = 0$ giant graviton brane

Following [11], we assume that the $Z = 0$ giant graviton brane has a boundary degree of freedom and full q-deformed $su(2|2)$ symmetry. We therefore introduce a boundary operator with an index B_j,

$$A_i^\dagger(p) B_j = R_{i\,j}^{i'j'}(p) A_{i'}^\dagger(-p) B_{j'},\qquad(4.16)$$

and we arrange the boundary S-matrix elements into the 16×16 matrix R,

$$R = R_{i\,j}^{i'j'} e_{i\,i'} \otimes e_{j\,j'}.\qquad(4.17)$$

It satisfies the right BYBE (cf. eq. (4.9))

$$S_{12}(p_1, p_2)R_{13}(p_1)S_{21}(p_2, -p_1)R_{23}(p_2) = R_{23}(p_2)S_{12}(p_1, -p_2)R_{13}(p_1)S_{21}(-p_2, -p_1).\qquad(4.18)$$

The vacuum state $B_j|0\rangle$ must form a fundamental representation of the symmetry algebra. The nontrivial actions of the Cartan generators are therefore given by (cf. eq. (3.4))

$$\begin{aligned} &h_1 B_1 = -B_1, && h_1 B_2 = B_2, && h_3 B_3 = -B_3, && h_3 B_4 = B_4,\\ &h_2 B_1 = -\left(C_B - \frac{1}{2}\right) B_1, && h_2 B_2 = -\left(C_B + \frac{1}{2}\right) B_2,\\ &h_2 B_3 = -\left(C_B - \frac{1}{2}\right) B_3, && h_2 B_4 = -\left(C_B + \frac{1}{2}\right) B_4. \end{aligned}\qquad(4.19)$$

The remaining such actions are trivial, $h_j B_k = 0$. The nontrivial actions of the bosonic simple roots are given by (cf. eq. (3.5))

$$E_1 B_1 = q^{1/2} B_2, \quad E_3 B_4 = q^{-1/2} B_3, \quad F_1 B_2 = q^{-1/2} B_1, \quad F_3 B_3 = q^{1/2} B_4,\qquad(4.20)$$

and the remaining such actions are trivial, $E_j B_k = F_j B_k = 0$. Finally, the nontrivial actions of the fermionic generators are given by (cf. eq. (3.7))

$$E_2 B_2 = a_B B_4, \quad E_2 B_3 = b_B B_1, \quad F_2 B_1 = c_B B_3, \quad F_2 B_4 = d_B B_2,\qquad(4.21)$$

and the remaining such actions are trivial. The vacuum state indeed forms a representation of the algebra (2.3) provided the parameters a_B, b_B, c_B, d_B obey the constraints

$$a_B d_B = \left[C_B + \frac{1}{2} \right]_q , \quad b_B c_B = \left[C_B - \frac{1}{2} \right]_q , \qquad (4.22)$$

which imply

$$(a_B d_B - q b_B c_B)(a_B d_B - q^{-1} b_B c_B) = 1 , \qquad (4.23)$$

in parallel with the bulk case.

A further important constraint on the parameters a_B, b_B, c_B, d_B comes from the requirement that the central charges P and K (2.7) commute with reflection from the boundary. Acting with P on both sides of

$$A_1^\dagger(p) \, B_1 \, |0\rangle = R_{1\,1}^{1\,1}(p) \, A_1^\dagger(-p) \, B_1 \, |0\rangle , \qquad (4.24)$$

we obtain

$$q^{C_B} e^{-ip} a(p) b(p) + q^{-C} e^{-ip} a_B b_B = q^{C_B} e^{ip} a(-p) b(-p) + q^{-C} e^{ip} a_B b_B , \qquad (4.25)$$

which implies

$$a_B b_B = -g q^{C_B} . \qquad (4.26)$$

Similarly, acting with K on (4.24), we obtain

$$c_B d_B = -g q^{C_B} . \qquad (4.27)$$

The constraints (4.26) and (4.27) are satisfied if we set

$$
\begin{aligned}
a_B &= \sqrt{g} \, \gamma_B \, q^{C_B/2} , \\
b_B &= -\sqrt{g} \, \frac{1}{\gamma_B} \, q^{C_B/2} , \\
c_B &= -i\sqrt{g} \, \gamma_B \, \frac{q^{(C_B+1)/2}}{x_B} , \\
d_B &= -i\sqrt{g} \, \frac{1}{\gamma_B} q^{(C_B-1)/2} x_B ,
\end{aligned}
\qquad (4.28)
$$

where γ_B is left unspecified.

The constraints (4.22) then imply

$$q^{2C_B} = \frac{1}{q} \left[1 + ig(q - q^{-1}) x_B/q \right]^{-1} = q \left[1 - ig(q - q^{-1})q/x_B \right]^{-1} . \qquad (4.29)$$

These relations in turn imply the quadratic constraint

$$x_B + \frac{1}{x_B} = \frac{i}{g} , \qquad (4.30)$$

which coincides with the result for the undeformed case [11].

Having specified the representation of the boundary operator, we can now proceed to determine the boundary S-matrix as before. The nonzero matrix elements are

$$R_{a\,a}^{a\,a} = \mathcal{A}\,, \qquad\qquad\qquad\qquad R_{\alpha\,\alpha}^{\alpha\,\alpha} = \mathcal{D}\,,$$

$$R_{a\,b}^{b\,a} = \frac{\mathcal{A} - \mathcal{B}}{q + q^{-1}}\,, \qquad\qquad R_{a\,b}^{a\,b} = \frac{q^{-\epsilon_{ab}}\mathcal{A} + q^{\epsilon_{ab}}\mathcal{B}}{q + q^{-1}}\,,$$

$$R_{\alpha\,\beta}^{\beta\,\alpha} = \frac{\mathcal{D} - \mathcal{E}}{q + q^{-1}}\,, \qquad\qquad R_{\alpha\,\beta}^{\alpha\,\beta} = \frac{q^{-\epsilon_{\alpha\beta}}\mathcal{D} + q^{\epsilon_{\alpha\beta}}\mathcal{E}}{q + q^{-1}}\,,$$

$$R_{a\,b}^{\alpha\,\beta} = -q^{(\epsilon_{ab}+\epsilon_{\alpha\beta})/2}\epsilon_{ab}\epsilon^{\alpha\beta}\frac{\mathcal{C}}{q + q^{-1}}\,, \qquad R_{\alpha\,\beta}^{a\,b} = q^{(\epsilon_{\alpha\beta}+\epsilon_{ab})/2}\epsilon^{ab}\epsilon_{\alpha\beta}\frac{\mathcal{F}}{q + q^{-1}}\,,$$

$$R_{a\,\alpha}^{a\,\alpha} = \mathcal{K}\,, \qquad\qquad\qquad R_{a\,\alpha}^{a\,\alpha} = \mathcal{L}\,, \quad R_{\alpha\,a}^{a\,\alpha} = \mathcal{G}\,, \quad R_{\alpha\,a}^{\alpha\,a} = \mathcal{H}\,, \qquad (4.31)$$

where $a\,,b \in \{1\,,2\}$ with $a \neq b$; and $\alpha\,,\beta \in \{3\,,4\}$ with $\alpha \neq \beta$; and

$$\mathcal{A} = \frac{\gamma(p)}{\gamma(-p)}\frac{qx^+ + x_B}{x^+(q - x_B s(x^-))}\,,$$

$$\mathcal{B} = \frac{\gamma(p)}{\gamma(-p)}\frac{x^- x^+(x_B + qx^+) + (1 + q^{-2})q^{-2C}((x^+)^2 - (qx^-)^2)(qx_B - x^+)}{x^-(x^+)^2(q - x_B s(x^-))}\,,$$

$$\mathcal{C} = \frac{q^{-5C/2}(1 + q^{-2})\gamma_B\gamma(p)\left[qx_B x^+(1 + x^- s(x^+)) + (qx^-)^2 - (x^+)^2\right]}{x^- x^+(q - x_B s(x^-))}\,,$$

$$\mathcal{D} = 1\,,$$

$$\mathcal{E} = \frac{1}{x^- x^+ s(x^+)(q - x_B s(x^-))}\Big\{s(x^-)\left[q^3(x^-)^2(1 + qx_B s(x^-)) + (1 + q^2)x_B x^+\right]$$
$$-(1 + q^{-2})q^{-2C}s(x^+)\left[qx_B x^+(1 + x^- s(x^+)) + (qx^-)^2 - (x^+)^2\right]\Big\}\,,$$

$$\mathcal{F} = \frac{1}{\gamma_B\gamma(-p)}\frac{q^{-3C/2}(1 + q^{-2})}{x^+(x^-)^2(q - x_B s(x^-))}\Big\{x^-\left[qx_B x^+(1 + x^- s(x^+)) + (qx^-)^2 - (x^+)^2\right]$$
$$-q^{2C}x_B\left[q^2(x^-)^3 s(x^+) + (x^+)^2\right]\Big\}\,,$$

$$\mathcal{G} = \frac{\gamma_B}{\gamma(-p)}\frac{q^{(C-1)/2}((x^+)^2 - q^2(x^-)^2)}{x^+ x^-(q - x_B s(x^-))}\,,$$

$$\mathcal{H} = \frac{qx^+ - x_B x^- s(x^+)}{qx^-(q - x_B s(x^-))}\,,$$

$$\mathcal{K} = \frac{\gamma(p)}{\gamma(-p)}\frac{q^3(x^-)^2 + x_B x^+}{qx^+ x^-(q - x_B s(x^-))}\,,$$

$$\mathcal{L} = \frac{\gamma(p)}{\gamma_B}\frac{q^{-(C+1)/2}x_B(1 + x^- s(x^+))}{x^-(q - x_B s(x^-))}\,. \qquad (4.32)$$

We have again set the overall scalar factor equal to one. We have explicitly verified with Mathematica that the BYBE (4.18) is satisfied. The singularity, which in the undeformed case is at $x^- = x_B$, is now given by $s(x^-) = q/x_B$.

5. Discussion

We constructed the q-deformation of the ZF formalism developed in [7, 18], which is con-

venient for performing explicit computations. We used this formalism to reobtain the bulk S-matrix of Beisert and Koroteev (3.19), (3.20). We determined how $x^\pm$ transforms under the reflection $p \mapsto -p$ in the q-deformed theory (4.1), and we found q-deformations of the $Y = 0$ and $Z = 0$ giant graviton brane boundary S-matrices of Hofman and Maldacena, namely, (4.15) and (4.31), (4.32), respectively.

It would be interesting to find additional boundary S-matrices, depending perhaps on one or more boundary parameters, by looking for linear combinations of generators which are preserved by the boundary. Indeed, the ZF formalism is well-suited for addressing that problem. As already mentioned in the Introduction, another interesting problem is to construct and solve open deformed Hubbard models based on the new boundary S-matrices. Finally, pursuing the speculation in [19] regarding a possible "$\mathrm{AdS}_q/\mathrm{CFT}_q$" duality, we simply note that our q-deformed boundary S-matrix could then describe the scattering of excitations of an open string attached to a quantum-deformed giant graviton in S_q^5.

Acknowledgments

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